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Gravity in a box

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Abstract: We consider a brane-world construction which incorporates a finite region of flat space, "the box", surrounded by a region of anti-de Sitter space. This hybrid construction provides a framework which interpolates between the scenario proposed by Arkani-Hamed, Dimopoulos and Dvali, and that proposed by Randall and Sundrum. Within this composite framework, we investigate the effects of resonant modes on four-dimensional gravity. We also show that, on a probe brane in the anti-de Sitter region, there is enhanced production of on-shell nonresonant modes. We compare our model to some recent attempts to incorporate the Randall-Sundrum scenario into superstring theory.

Keywords: Extra Large Dimensions, Classical Theories of Gravity.

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Contents

1. Introduction

There has been a renewed interest by the high energy community in the possibility that spacetime may have a dimension larger than four. The essential new concept which sparked this interest is the notion that only gravitational excitations propagate through the full spacetime, while all of the observed Standard Model particles are confined to a three-brane. Such a scenario may be motivated by ideas naturally arising in string theory [1, 2]. One of the most exciting phenomenological implications of these brane-world scenarios is that the fundamental scale of gravity may be reduced from the four-dimensional Planck scale of 10^{16} TeV to as little as 1 TeV.

The wave of research exploring these scenarios has moved forward on two seemingly disjoint fronts. In the original work of Arkani-Hamed et al. [3, 4], there are large extra dimensions, which may be as big as a fraction of a millimeter. For simplicity, the extra compact dimensions are usually assumed to be toroidal,[1] and then the gravitational fluctuations have a simple Fourier mode expansion. The zero-mode in this expansion is interpreted as the massless four-dimensional graviton responsible for the observed long-range effects of gravity.

[1] Compactifications using hyperbolic manifolds were considered in ref. [5].

More recently, Randall and Sundrum [6, 7] proposed a scheme in which a five-dimensional spacetime contains strongly gravitating three-branes which then produce a warped or nonfactorizable geometry. With a particular tuning of the bulk (negative) cosmological constant and the brane tension, the induced geometry on the three-branes is just flat four-dimensional Minkowski space.[2] There is also a massless four-dimensional graviton, but its wave function is *localized* in the nonfactorizable geometry of the extra dimension. In this construction, the size of the extra dimension is unconstrained: it could be either very small [6] or very large [7]. This scenario can also be extended to cases with more than one extra dimension [9].

In this paper, we study a brane setup which interpolates between these two classes of scenarios, with a single extra dimension. Our five-dimensional construction contains a nonfactorizable geometry which is asymptotically anti-de Sitter (AdS), as in the Randall-Sundrum (RS) scenario. However, as in the framework of Arkani-Hamed-Dimopoulos-Dvali (ADD), there is a completely flat region of finite width — "the box" — bounded by three-branes.[3] By varying the relative scales of the flat box and the AdS regions, this setup interpolates between a limit in which it reproduces the RS construction and another where it yields the ADD scenario. We will study the low-energy effective theory of four-dimensional gravity that arises on the branes. In particular, we will focus on the effects of "resonant modes", which are metric modes which have enhanced support inside the box region.

2. Background geometry

We begin by considering a five-dimensional nonfactorizable background geometry whose metric takes the following form in Poincaré coordinates:

$$ds^2 = e^{-2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2\,. \tag{2.1}$$

For simplicity, we will restrict our attention to geometries (both the background and the metric fluctuations in the following section) which are reflection symmetric around $y = 0$. Thus we are essentially considering a Z_2 orbifold (which may be either compact or noncompact). We will also be interested in the case where the geometry is asymptotically AdS, i.e. for large $|y|$,

$$A(y) \to k|y|\,, \tag{2.2}$$

where k is the inverse of the AdS radius of curvature.

[2]One can also induce cosmological geometries on the three-branes by varying these parameters [8].

[3]An interesting construction based on the opposite hybrid picture, where a finite AdS region is surrounded by an asymptotically flat space, was recently proposed [10]. Whether this yields a viable effective theory of four-dimensional gravity is currently under scrutiny [11].

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The simplest example of a brane setup which produces a background geometry of this type is to have a number of three-branes with positive tension superimposed at $y = 0$. We refer to these branes collectively as the "Planck brane", and we will designate this simple setup as the *RS limit* — see below. We consider the five-dimensional gravity action

$$
\begin{aligned}
S &= S_{\rm bulk} + S_{\rm brane}\,, \\
\text{with} \qquad S_{\rm bulk} &= \int d^4x\, dy\, \sqrt{-g}\,(2M_5^3 R - \Lambda)\,, \\
S_{\rm brane} &= -\int d^4x\, \sqrt{-g_4}\, V_P\,,
\end{aligned}
\tag{2.3}
$$

which should be considered the leading terms in a low-energy effective action. Here, g_{MN} $(M, N = 0, \ldots 4)$ is the five-dimensional metric, while $(g_4)_{\mu\nu}$ $(\mu, \nu = 0, \ldots 3)$ is the induced metric on the Planck brane. Also, M_5, Λ and V_P denote the five-dimensional Planck scale, the (negative) bulk cosmological constant and the (total) brane tension, respectively. Finding a solution which is Poincaré invariant in four dimensions requires that the tension V_P is tuned relative to the cosmological constant Λ. That is, we set $V_P = \sqrt{24M_5^3|\Lambda|}$, as in [6]. The solution of the five-dimensional Einstein equations in this RS setup is then given by the metric (2.1), with $A(y) = k|y|$, where

$$
k^2 = -\frac{\Lambda}{24M_5^3}\,. \tag{2.4}
$$

This background geometry is simply two AdS regions glued together along the surface $y = 0$ with the Planck brane supporting the appropriate discontinuity in the extrinsic curvature across the gluing surface.

Now imagine splitting the Planck brane into two sets and pulling them away symmetrically from $y = 0$ to $y = \pm y_0$. Given a symmetric division of the Planck brane, the tension of each of the two subsets is $V_P/2$. We also require that the region of the bulk space between the branes (i.e. with $|y| < y_0$) is in a new vacuum where the bulk cosmological constant vanishes. Then the three-branes at $y = y_0$ remain flat with the same tuning of the tension V_P given above. Given the vanishing of Λ in the small y region, solving Einstein's equations in this part of the spacetime will yield a slice of flat five-dimensional Minkowski space. (The solution of extreme vacuum domain walls interpolating between AdS and Minkowski spacetime was studied in four dimensions by [12] and generalized to D-dimensions in [13].) The full solution becomes the metric (2.1) with

$$
A(y) = \frac{1}{2}k|y - y_0| + \frac{1}{2}k|y + y_0| - ky_0\,, \tag{2.5}
$$

where k has the same value as in eq. (2.4). The resulting picture is then a flat "box" glued between two AdS regions. Of course, the RS limit is now that in which the box shrinks to zero size, i.e. $y_0 \to 0$.

3. Graviton modes

When linearized metric fluctuations are included, the geometry takes the form

$$ds^2 = (e^{-2A(y)}\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dy^2 \,. \tag{3.1}$$

In the following, we will work in a gauge where $\partial^\mu h_{\mu\nu} = h^\mu_\mu = 0$. We will not consider the metric fluctuations h_{55} and $h_{5\mu}$ (which are pure gauge for the case where y has an infinite range). It is useful to define a conformal coordinate z by $z \equiv sgn(y)[(e^{k(|y|-y_0)} - 1)/k + y_0]$ when $|y| \geq y_0$, and $z \equiv y$ when $|y| \leq y_0$. Given our background solution (2.5), the geometry (3.1) then becomes

$$ds^2 = \begin{cases} (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dz^2 & \text{for } |z| \leq z_0 \\ \dfrac{1}{(k\tilde{z})^2}\left[(\eta_{\mu\nu} + \hat{h}_{\mu\nu})dx^\mu dx^\nu + dz^2\right] & \text{for } |z| \geq z_0 \end{cases} \tag{3.2}$$

where $z_0 = y_0$, $\tilde{z} = |z| - z_0 + 1/k$, and $\hat{h}_{\mu\nu} = e^{2A(z)}h_{\mu\nu}$.

Now solve the linearized Einstein equations for $h_{\mu\nu}$ with separation of variables using an ansätz of the form: $h_{\mu\nu} = e^{ip\cdot x}e^{-A(z)/2}\psi_m(z)\epsilon_{\mu\nu}$. Here $\epsilon_{\mu\nu}$ is a constant polarization tensor. The four-dimensional profile of these solutions is a plane wave with an effective four-dimensional mass: $m^2 = -p^2$. Solving the linearized equations is now reduced to a one-dimensional Schrödinger problem:

$$\left[-\frac{1}{2}\partial_z^2 + V(z)\right]\psi_m(z) = \frac{1}{2}m^2\psi_m(z)\,, \tag{3.3}$$

where the potential $V(z)$ is given by

$$V(z) = \frac{15k^2}{8(k|z| - kz_0 + 1)^2}\theta(|z| - z_0) - \frac{3k}{4}\delta(|z| - z_0)\,. \tag{3.4}$$

With these definitions, the natural norm[4] for the profile in the fifth dimension is simply $\int dz\, |\psi_m(z)|^2 = 1$.

The solution to eq. (3.3) is a combination of plane waves in the box and Bessel functions in the AdS region:

$$\psi_m(z) = \begin{cases} B_m \cos mz & \text{for } |z| < z_0\,, \\ N_m(k\tilde{z})^{1/2}\left[Y_2(m\tilde{z}) + L_m\, J_2(m\tilde{z})\right] & \text{for } |z| > z_0\,, \end{cases} \tag{3.5}$$

where $\tilde{z}$ is as defined below eq. (3.2), while B_m, N_m, and L_m are m-dependent coefficients. L_m is determined by the jump condition at the Planck brane,

$$L_m = -\frac{Y_1(m/k) + Y_2(m/k)\,\tan(mz_0)}{J_1(m/k) + J_2(m/k)\,\tan(mz_0)}\,, \tag{3.6}$$

[4]This norm is inherited from the standard relativistic or "Klein-Gordon" inner product of the five-dimensional graviton fluctuations in the given background geometry (2.1).

while B_m is fixed by requiring the ψ_m to be continuous at z_0,

$$B_m \cos(mz_0) = N_m \left[Y_2(m/k) + L_m\, J_2(m/k)\right]. \tag{3.7}$$

This leaves N_m to be fixed by imposing the appropriate normalization of the profile.

We can introduce a second boundary, i.e., a second Z_2 orbifold surface, at some finite $z = z_c$ in the AdS region, by inserting branes of negative tension $V_N = -V_P$, as in [6]. In this case, the mass spectrum becomes discrete because a second jump condition must be imposed on the mode functions at the negative tension branes. The latter provides an independent equation fixing L_m, which combined with eq. (3.6) yields

$$\frac{Y_1(m/k) + Y_2(m/k)\tan(mz_0)}{J_1(m/k) + J_2(m/k)\tan(mz_0)} = \frac{Y_1(m\Delta z + m/k)}{J_1(m\Delta z + m/k)}\,, \tag{3.8}$$

where $\Delta z = z_c - z_0$.

If we take $\Delta z \to 0$, the two sets of branes “annihilate” leaving behind a braneless compactification on S^1/Z_2. This is a smooth limit in our low-energy description. We will refer to this as the *ADD limit*. For this limit to be phenomenologically viable, one would have to assume that there are Standard Model fields living on the orbifold surfaces, in analogy to the M-theory scenario proposed in [1]. Alternatively, one could introduce a probe brane (which does not disturb the background geometry) to support the Standard Model fields.

In all of the cases that we are considering, there is a normalizable zero-mode:

$$\psi_0(z) = \begin{cases} B_0 & \text{for } |z| < z_0\,, \\ B_0\,(k\tilde{z})^{-3/2} & \text{for } |z| > z_0\,. \end{cases} \tag{3.9}$$

The normalization condition determines B_0:

$$B_0 = \left(\frac{k}{2kz_0 + 1 - e^{-2k\Delta z}}\right)^{1/2}. \tag{3.10}$$

where the last term in the denominator vanishes in the case of an infinite fifth dimension, i.e. when $\Delta z \to \infty$.

The existence of this zero-mode is also evident as follows: One finds that the five-dimensional metric (2.1) remains a solution of the field equations derived from eq. (2.3) when the flat metric $\eta_{\mu\nu}$ is replaced by a general Ricci-flat metric $\tilde{g}_{\mu\nu}(x)$. That is, the five-dimensional equations of motion are still satisfied as long as the brane metric satisfies the four-dimensional Einstein equations $R_{\mu\nu}(\tilde{g}) = 0$. The zero-mode solutions appearing in the linearized calculations above are the usual gravity waves appearing in a perturbative analysis of these four-dimensional gravity equations.

Using this general nonlinear ansätz, we can also calculate the effective four-dimensional Planck scale for observers on the three-branes at $z = z_0$. We simply insert our ansätz, eqs. (2.1) and (2.5), with $\tilde{g}_{\mu\nu}(x)$ replacing $\eta_{\mu\nu}$, into the five-dimensional action (2.3). Now integrating over y leaves an effective four-dimensional

Einstein action with an overall coefficient of $2M_{\text{Planck}}^2$ where

$$M_{\text{Planck}}^2 = \frac{M_5^3}{B_0^2} = \frac{M_5^3}{k}\left(2kz_0 + 1 - e^{-2k\Delta z}\right). \tag{3.11}$$

Again, the final term in the second factor vanishes for the case of an infinite fifth dimension.

4. An infinite fifth dimension

First we consider the case of the box embedded in an AdS space without boundary. There is a continuum spectrum of massive gravity modes. Unlike the zero-mode (3.9), a unit norm can not be imposed on these modes because the $\psi_m(z)$ have plane wave behavior asymptotically. Instead these modes are given δ-function normalization, i.e. $\int \psi_m^*(z)\,\psi_{m'}(z)\,dz = \delta(m-m')$. Comparing to eq. (3.5), this implies:

$$N_m^2 = \frac{1}{2}\frac{m}{k}\frac{1}{1+L_m^2}. \tag{4.1}$$

The coefficient B_m is obtained from eq. (3.7).

Now the essential question we would like to answer is the extent to which gravity on the branes at $z = z_0$ is four-dimensional. In particular, we will examine how the gravitational potential is modified by the massive gravity modes in the bulk. Following [7],[5] the gravitational potential between two test masses, m_1 and m_2, separated by a distance r on the Planck brane, takes the form

$$U(r) = -\frac{G_4 m_1 m_2}{r}\left(1 + \int dm\,\rho(m)\,e^{-mr}\right). \tag{4.2}$$

Here the four-dimensional Newton's constant is defined as

$$G_4 \equiv \frac{1}{32\pi M_{\text{Planck}}^2} = \frac{1}{32\pi M_5^3}\frac{k}{2kz_0+1}. \tag{4.3}$$

We have also introduced a relative density of states

$$\rho(m) \equiv \frac{|\psi_m(z_0)|^2}{|\psi_0(z_0)|^2} \tag{4.4}$$

for the massive modes. Now for large distances, the dominant contributions to the integral over the massive modes will come from $m < 1/r$. In general, we only consider distances $r > 1/k$ where these dominant contributions come from $m < k$. This latter restriction is made because from the form of the potential (3.4) in the Schrödinger equation (3.3), it is clear that modes with $m \gtrsim k$ will not be suppressed at the

[5] see [14] for a more extensive discussion.

Planck brane. The strong coupling of these bulk modes indicates that we should expect that the approximately four-dimensional character of gravity on the brane must break down for $r < 1/k$.

Now if the size of the box is small or comparable to the AdS scale, i.e., $kz_0 \lesssim 1$, it is not hard to show that the leading corrections to the long-range gravitational potential are in fact identical to those in the RS limit. One finds

$$\rho(m) \simeq \frac{1}{2}\frac{m}{k^2} \tag{4.5}$$

and so is independent of z_0. The final result for the gravitational potential is

$$U(r) \simeq -\frac{G_4 m_1 m_2}{r}\left(1 + \frac{1}{2k^2r^2}\right). \tag{4.6}$$

Hence there is a power law correction to the four-dimensional newtonian potential, which is controlled by the AdS scale k.

In the regime of a large box, i.e., $kz_0 \gg 1$, more interesting behavior is found. In figure 1, we plot L_m and $|B_m|$ as a function of mz_0 for fixed z_0. The figure illustrates the generic behavior in this regime. That is, B_m goes through periodic extrema as m increases, while at these m values L_m is very close to 0. These modes at the extrema of B_m, which have enhanced support inside the box, are identified as the *resonant modes*. Numerically, we find that these resonances persist for relatively small boxes, e.g., $kz_0 \simeq 10$, but to produce precise analytic results below, we will restrict our attention to the regime $kz_0 \gg 1$.

In our long-range or low-energy approximation, we restrict our attention to light modes: $m/k \ll 1$. The condition $L_m \sim 0$ for resonant modes then reduces to

$$\tan mz_0 \simeq -\frac{m}{2k}, \tag{4.7}$$

which means that $mz_0 \simeq n\pi$ for some positive integer n. Near the zero, the tangent function is essentially linear, so to leading order we can write

$$\tan mz_0 \simeq mz_0 - n\pi \tag{4.8}$$

and we find

$$L_m \simeq \frac{4}{\pi}\frac{k^3}{m^3}(x - 2\pi n), \tag{4.9}$$

where $x \equiv (2kz_0 + 1)(m/k)$. Thus the resonance mass is

$$m_n \simeq \frac{2\pi nk}{2kz_0 + 1} \simeq \frac{\pi n}{z_0}. \tag{4.10}$$

Now near the resonant masses, the value of the wave functions on the Planck brane can be written as

$$|\psi_m(z_0)|^2 \simeq |B_m|^2 \simeq \left|N_m Y_2\left(\frac{m}{k}\right)\right|^2 \simeq \frac{2}{\pi}\frac{Q}{1 + Q^2(x - 2\pi n)^2}, \tag{4.11}$$

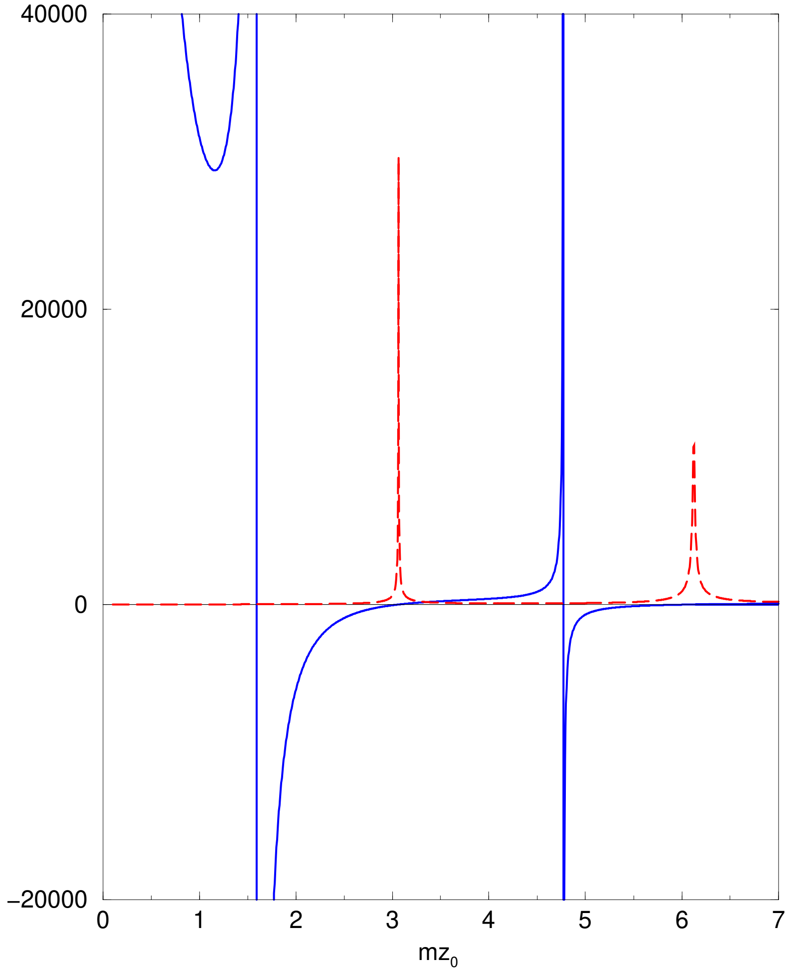

Figure 1: L_m (solid line) and $|B_m|$ (dashed line) as functions of mz_0. $kz_0 = 20$ is chosen for the plot, and B_m is rescaled by a factor of 2000.

where

$$Q \equiv \frac{4}{\pi}\left(\frac{k}{m}\right)^3 . \tag{4.12}$$

Note that the extremal value of $|\psi_m(z_0)|^2$ is proportional to Q, while the width at half-maximum is $\Delta x = 2Q$. Hence from eq. (4.12), one sees that the peaks in eq. (4.11) become higher and narrower for smaller resonant masses — a feature which can be observed in the plot of $|B_m|$ in figure 1. With $m/k \ll 1$, Q is large and the expression in eq. (4.11) is a good approximation to 2 times a delta function. Hence the correction to the Newtonian potential (4.2) becomes

$$\begin{aligned}\int dm\, \rho(m)\, e^{-mr} &\simeq \frac{2}{\pi}\sum_{n=1}^{\infty}\int dx \frac{Q}{1+Q^2x^2}e^{-m_n r}\\ &= 2\sum_{n=1}^{\infty} e^{-m_n r}\,,\end{aligned} \tag{4.13}$$

where we have treated both Q and m in the exponential as slowly varying functions. We have also used $dm = |\psi_0(z_0)|^2 dx$. The sum over n is actually cut off at $n \sim kz_0$.

Within our approximations then, the continuum of states around a resonant mass makes a contribution to the newtonian potential as though there were a single *discrete* normalizable mode with mass $m = m_n$. The total gravitational potential on the Planck brane becomes

$$U(r) \simeq -\frac{G_4 m_1 m_2}{r}\left(1 + \frac{1}{2k^2r^2} + 2\frac{e^{-r/r_e}}{1-e^{-r/r_e}}\right), \tag{4.14}$$

where the second term comes from the nonresonant continuum modes, and the third term comes from summing over the contributions at all of the resonant masses. The effective length scale appearing in this last term is: $r_e \simeq z_0/\pi$. Therefore within the large box regime with $kz_0 \gg 1$, these resonant mode contributions are in fact the leading contributions to the newtonian gravitational potential.

5. A finite Z_2 orbifold

In the case where a second orbifold surface is introduced at $z = z_c$, the spectrum of the gravity modes becomes discrete as determined by eq. (3.8). Hence the details of the spectrum are controlled by the three different scales, z_0, Δz, and $1/k$, entering this quantization constraint.

First consider the regime: $\Delta z \gg z_0, 1/k$. In this case the spacing of the masses is very small, $\delta m \simeq \pi/\Delta z$, which can be seen as follows: on the right-hand-side (r.h.s.) of eq. (3.8), we can approximate the Bessel functions with their asymptotic plane wave forms to give

$$\frac{Y_1(m/k) + Y_2(m/k)\tan(mz_0)}{J_1(m/k) + J_2(m/k)\tan(mz_0)} \simeq \tan\left(m\Delta z + \frac{m}{k} - 3\frac{\pi}{4}\right). \tag{5.1}$$

Now the r.h.s. is a rapidly varying function of m compared to the l.h.s. In particular, the r.h.s. varies from $-\infty$ to ∞ as m increases by $\pi/\Delta z$. Hence this constraint (5.1) will be satisfied once in every interval $n\pi/\Delta z < m < (n+1)\pi/\Delta z$. With such a tight spacing of the mass spectrum, the physics is still essentially unchanged from the case with an infinite fifth dimension discussed in the previous section. In particular, if we are also in the regime where $z_0 \gg 1/k$, the modes in the discrete spectrum satisfying $m \simeq n\pi/z_0$ will have enhanced support in the box region. These resonant modes will then dominate the corrections to the four-dimensional gravitational potential.

Let's now consider the situation in which the size of the box is much larger than the size of the AdS space, i.e. $z_0 \gg \Delta z \gg 1/k$. For modes with $m \ll k$, eq. (3.8) gives:

$$\tan(mz_0) \simeq -\frac{m}{2k} \ll 1\,, \tag{5.2}$$

thus $m \sim n\pi/z_0$ and $\cos(mz_0) \sim 1$. Note that this result is precisely the condition for the resonances (4.7) in the case of the infinite fifth dimension.

With $m\tilde{z} \ll 1$, Y_2 dominates the shape of the wave function outside the box. The normalization condition can be written $I_1 + I_2 = 1$, where I_1 is the contribution from outside of the box:

$$I_1 \simeq kN_m^2 \int_{z_0}^{z_0+\Delta z} dz\, \tilde{z}\, |Y_2(m\tilde{z})|^2 \simeq N_m^2\, \frac{16}{\pi^2}\, \frac{k^4}{m^4}\, \frac{1}{k}\,, \tag{5.3}$$

and I_2 is the contribution from inside the box:

$$\begin{aligned} I_2 &\simeq B_m^2 z_0\,, \\ &\simeq N_m^2\, \frac{16}{\pi^2}\, \frac{k^4}{m^4}\, z_0\,. \end{aligned} \tag{5.4}$$

Since we are considering $z_0 \gg 1/k$, $I_2 \gg I_1$ and $B_m \simeq 1/\sqrt{z_0}$.

Therefore, $\rho(m) \simeq 1/2$ on the Planck brane. Thus up to a factor of order one, the leading correction to the gravity potential is the same as the third term in eq. (4.14), with the same effective length scale (given the matching between eqs. (4.7) and 5.2). Hence in this regime, the large box again mimics the situation with one extra flat dimension of size z_0.

6. No unusually large boxes

In the case just described the size z_0 of the box cannot be larger than a millimeter without conflicting with Cavendish type experiments which directly measure the newtonian potential. Now we pause to inquire whether it is possible to weaken this limit, by somehow introducing a large wave function suppression for the resonant modes.

One obvious strategy is to change the location of our observer in the fifth dimension; to this end we could confine the Standard Model to a probe brane, as in the scenario of Lykken and Randall [15]. However locating the probe brane in the interior of the box is no help, and locating it in the interior of the AdS region is no better, since for light modes, Y_2 dominates the wave function and simply tracks the zero-mode. Thus $\rho(m) \sim 1$ still applies.

The difference between the modes in eq. (5.2) and that of RS is that in the first case the behavior of these resonant modes is dominated by function Y_2 in both the small *and* large z regions of AdS, while in the RS case, ψ_m is dominated by Y_2 when it is close to the Planck brane, but by J_2 when it is far away. The suppression of the massive graviton wave functions on the Planck brane in the RS limit comes as a balance between the normalization factor, determined mainly by J_2 term, and the behavior of Y_2 at the Planck brane.

However, one could imagine another situation where ψ_m is dominated by J_2 thoughout the AdS region. This would cause suppression of the massive graviton wave function on the Planck brane since $J_2(m/k) \sim (m/k)^2$ which is small for the light modes. We have found that there indeed exist such modes if one assumes that $\Delta z \sim z_0 \gg 1/k$.

If $m\Delta z$ is large enough such that Y_1 and J_1 are in their asymptotic region,[6] while $m/k \ll 1$ is still satisfied, the l.h.s. of eq. (3.8) becomes $\tan(m\Delta z - 3\pi/4)$. Both sides of eq. (3.8) can be large provided that:

$$\tan(mz_0) \sim -\frac{J_1(m/k)}{J_2(m/k)} \sim -\frac{4k}{m}\,, \tag{6.1}$$

and

$$\tan\left(m\Delta z - \frac{3\pi}{4}\right) \gg 1\,, \tag{6.2}$$

which implies that

$$mz_0 \simeq \frac{\pi}{2}(2n+1)\,, \tag{6.3}$$

$$m\Delta z \simeq \frac{\pi}{2}(2l+1) + \frac{3\pi}{4}\,, \tag{6.4}$$

where the integers n and l can be different, depending on the ratio between Δz and z_0.

The normalization contribution to the wave function from the AdS region takes the following form for $m\Delta z$ being large ($\gtrsim 5$):

$$I_1 \simeq \frac{N_m^2 L_m^2}{\pi}\frac{k}{m}\Delta z\,. \tag{6.5}$$

The normalization contribution from the box gives

$$I_2 \simeq B_m^2 z_0 \simeq \frac{N_m^2 L_m^2}{4}\frac{m^2}{k^2} z_0\,, \tag{6.6}$$

where $\tan(mz_0) \sim -4k/m$ from the quantization condition has been applied. It is obvious that in the limit $\Delta z \sim z_0 \gg 1/k$, I_1 is always much larger than I_2 for light modes ($m \ll k$), thus the AdS behavior dominates the normalization.

One can then calculate the ratio $|\psi_m|^2/|\psi_0|^2$ on the Planck brane,

$$\rho(m) \simeq \frac{2\pi}{64}\frac{mz_0}{k\Delta z}\frac{m^4}{k^4} \simeq \mathcal{O}\left(\frac{m^5}{k^5}\right). \tag{6.7}$$

For these modes there is indeed an enormous suppression at the Planck brane!

Unfortunately, it is not possible to adjust parameters such that *all* massive modes are suppressed on the Planck brane. By choosing different $z_0/\Delta z$ as implied by eq. (6.4), we have changed the periodicity of the Bessel functions, i.e., the r.h.s. of the eq. (3.8) as a function of (mz_0) has a periodicity of $\pi z_0/\Delta z$, while the l.h.s. is approximately periodic in (mz_0) with periodicity π. As a result, between two adjacent values mz_0 given by eq. (6.4) which satisfy eq. (3.8) and yield large L_m, the l.h.s. and the r.h.s. of eq. (3.8) meet many times, and inevitably some of these solutions yield modes with small L_m. As we discussed earlier, when L_m is small, ($L_m \ll (k/m)^4$), the mode is unsuppressed at the Planck brane.

[6] $\sqrt{t}J_1(t)$ is very close to its asymptotic form $\sqrt{2/\pi}\cos(t - 3\pi/4)$ at $t \geq 3$, while the small argument behavior $J_1(t) \sim t^2/8$ is a good approximation with $t \leq 1.2$. The first few zeros of $J_1(t)$ are at $t = 3.83$, 7.01, 10.17, 13.32, 16.47.

7. Collider phenomenology

The typical cross sections for the total production of on-shell massive gravitons on the Planck brane is given by [3]:

$$\sigma \sim \frac{1}{M_{\text{Planck}}^2} \int_0^E dm\ \rho(m)\,, \tag{7.1}$$

up to dimensionless couplings and numerical factors. Here the integral over the density of states extends up to some maximum kinematically available energy scale E. We assume for simplicity an infinite AdS region, and restrict $\frac{m}{k} \ll 1$ over the entire integration region. For a large box, i.e. $kz_0 \gg 1$, the leading order contribution to the integral over the density of states comes from the resonant modes:

$$\int_0^E dm\, \rho(m) = \frac{2}{\pi} E z_0 + O\left(\frac{E}{k}\right)^3 . \tag{7.2}$$

This result of course mimics an ADD scenario with one extra dimension having the size of z_0. Even for a millimeter-size box (the largest allowed by the Cavendish type bounds), the z_0 wave function enhancement of the cross section cannot overcome the Planck suppression of the couplings. Thus if we live on the Planck brane there are no observable collider effects from on-shell production of bulk modes.

One obvious extension of our construction is to change the location of the Standard Model fields. Here we can imagine confining the Standard Model to a probe three-brane, as in the scenario of Lykken and Randall [15]. Locating the probe brane inside the box would not seem to lead to any new interesting physics since, as seen in eq. (3.5), the suppression (or enhancement) of the bulk modes levels off for $|z| < z_0$. Hence we consider locating the probe brane at some finite distance $\Delta z_p = z_p - z_0$ inside the infinite AdS region. Here one can profit from the AdS geometry to generate an interesting hierarchy of scales [6]. As shown in [15], the very light continuum bulk modes which contribute to the Newtonian potential on the brane have wave functions which simply track the zero-mode. In our case all of the resonant modes also track the zero-mode, so the complete result for the Newtonian potential on the probe brane is as given in eq. (4.14).

It was also shown in [15] that the continuum bulk modes which are heavier than $1/(k\Delta {z_p}^2)$ have wave functions at the probe brane which are dominated by the J_2 rather than Y_2 behavior. These modes do not track the zero-mode, and have large wave function enhancements, leading to potentially observable collider effects. This behavior still holds in the case we are considering, precisely for the nonresonant gravity modes, i.e. the modes which do not have $L_m \to 0$ and thus constitute the nonresonant portion of the bulk continuum. The total cross section for production of these modes on the probe brane is given by:

$$\sigma \sim (2kz_0 + 1) \frac{k^2}{M_{\text{Planck}}^2} E^6\, \Delta {z_p}^8 \tag{7.3}$$

where the result is presented for $k\Delta z_p \gg 1$, as well as $m/k \ll 1$. In the limit that $kz_0 \to 0$, this expression reduces to precisely that derived in [15]. However, the factor of $(2kz_0+1)$ in eq. (7.3) provides a relative enhancement for $kz_0 \gg 1$, i.e., for a large box. The appearance of this factor can be traced to a suppression of the zero-mode wave function (relative to that of the continuum bulk modes) in this scenario, as seen in eqs. (3.9) and (3.10). The same suppression then effects the definition of M_{Planck} in eq. (3.11).

The net result is that the existence of a large box can dramatically enhance the collider signals on a probe brane in the infinite AdS region. The enhancement is not due to resonant mode production (which as we have seen is Planck suppressed), but rather to the enhanced production of the continuum bulk modes. For example, suppose that $k \sim M_{\text{Planck}}$ and that we have a millimeter-sized box. Then kz_0 is a huge enhancement factor: $kz_0 \simeq 10^{16}$. This would imply that, even with a probe brane cutoff $1/\Delta z_p$ as large as $10^4\,\text{TeV}$, collider effects are suppressed by no more than $E^6/(\text{TeV})^8$. Hence, in the large box scenario, the production of these bulk modes could be within the reach of collider experiments in the forseeable future. One might also expect that this enhanced production may have observable astrophysical and cosmological implications.

8. Discussion

To summarize, we have presented a brane construction which smoothly interpolates between the physics of the ADD and RS scenarios. Essentially, our five-dimensional background geometry consists of a slice of flat Minkowski space, "the box", glued between two AdS regions. The discontinuity in the extrinsic curvature across the gluing surfaces is interpreted in terms of positive-tension three-branes located at these positions. For the most part in the following discussion, we will explicitly comment on the case of an infinite fifth dimension, however, most of the comments carry over to the case where the fifth dimension is finite.

If the size of the box is small compared to the AdS scale, i.e., $z_0 \lesssim 1/k$, then the low-energy physics is essentially the same as in the RS scenario. That is, the coupling of the bulk gravity modes is still essentially controlled by the AdS scale, and so the corrections to the gravitational potential (4.6) have precisely the same power law form as in [7]. Even if $z_0 \sim 1/k$, there would essentially be only one scale in the potential 3.4, and so one should not find any radical departures from the RS scenario.

This small box regime, i.e., $z_0 \lesssim 1/k$, would model the situation of a thick or smooth Planck brane [16]. That is a construction where one might attempt to realize the RS scenario using a smooth domain wall solution to replace the infinitely thin Planck brane. One would expect (at least naively) that since, in such a scenario, the AdS curvature and the thickness of the brane would be determined by the same underlying microscopic theory, both of these scales would be of the same order. Our results in the small box regime then agree with the investigations in [16], where it was found that thickening the Planck brane produced no significant differences from the low-energy physics of the RS scenario.

In the large box regime, i.e., $z_0 \gg 1/k$, a new large length scale is introduced and we do find significant changes in the low energy physics. In particular, there are resonant modes with enhanced support inside the box, and so with an enhanced coupling to the the three-branes. Even though there is a continuum of bulk modes with masses near the resonant mass, their net contribution mimics a single normalizable mode with this resonant mass. Thus the details of the AdS region are suppressed in this regime, and to leading order in $1/kz_0$ the brane-boundary at $z = z_0$ simply acts like an orbifold surface. That is, the leading order corrections to the Newtonian potential are identical to those as if we were considering a compactification of flat five-dimensional spacetime on S^1/Z_2.

Recently, H. Verlinde [17] proposed an interesting way to realize the RS scenario in superstring theory. This proposal was later elaborated on in [18]. Essentially the five-dimensional AdS geometry arises in the throat geometry near a cluster of D3-branes, while the five-form charge of the D-branes is absorbed by the "topology" of the compactification geometry in which they are positioned. The Standard Model fields would live on a probe brane sitting in the AdS throat, analogous to [15]. Like our "gravity-in-a-box" model, this scenario then has two independent scales, the AdS curvature scale $1/k$ and the size of the compactification manifold L. Further the latter size must be larger than the AdS scale in order that the throat of the D3-branes can fit inside this geometry. Hence we expect that our large box scenario may be closely related to the low-energy physics of Verlinde's construction. In particular then, as in eq. (3.11) with large kz_0, the relation between the observed four-dimensional Planck scale and the fundamental scale of gravity will be essentially the same as in the standard ADD scenario. This simple relation arises because the normalization of the graviton zero-mode is dominated by the integration over the volume of the flat box, external to the AdS region. As observed in section 7 then, the latter also results in the production rate of continuum bulk gravitons being enhanced. We expect this enhancement will be a general feature of the superstring constructions, and so provide interesting phenomenological constraints for these models

It is interesting to consider the generalization of our brane construction to spacetimes with more than five dimensions. The RS scenario was generalized to higher dimensions using intersecting branes in [9]. This discussion was extended to considering both intersecting branes and different cosmological constants in the distinct regions between the branes [19]. Given these results, it is clear that there is no obstacle to extending the present scenario to higher dimensions. One would have a finite portion of flat space surrounded by various AdS regions. If the size of the box is still characterized by a single scale, we expect that much of the previous discussion would carry over to the present situation. If the box is smaller than the curvature scale of the surrounding AdS regions, that the low-energy physics would be essentially the same as in generalized RS construction of [9]. If the size of the box is much bigger than the AdS scale, there should be resonant modes so that the low-energy theory imitates a flat space ADD scenario. We expect that just as the ADD scenarios are more phenomenologically interesting in more than five dimensions, the higher dimensional extensions of our "gravity-in-a-box" model would yield a richer phenomenology. It may be of interest to examine how to distinguish the low-energy physics of the ADD scenario from that of a large higher dimensional box. One interesting possibility for our higher dimensional constructions is that one can engineer a box with an essentially arbitrary shape in the extra dimensions. In the large box regime, such a configuration should give rise to a unique spectrum of masses which would distinguish it from a conventional ADD scenario.

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